ISRAEL JOURNAL OF MATHEMATICS, Vol. 73, No. 3, 1991

THE PRODUCT STRUCTURE OF FINITELY PRESENTED DYNAMICAL SYSTEMS

BY

NICOLAI T. A. HAYDN

Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA

ABSTRACT

We consider finitely presented systems, which were introduced by Fried, and examine the circumstances under which these systems have canonical coordinates. We give necessary and sufficient conditions for their existence in a combinatorial way.

Finitely presented systems which were introduced by D. Fried [9] with the intention of generalising symbolic description of dynamical systems have recently become of interest as a class of dynamical systems to which the extensive theory of Axiom A systems and sofic systems can be extended without incurring too many casualities. However, one property that gets lost in generalising is the local product structure or canonical coordinates, which characterise strongly hyperbolic systems: with it the shadowing property also goes, although, as Fried shows, there are still finite Markov partitions. On the other hand, much of the theory on Axiom A systems can indeed also be formulated for finitely presented systems. As an example we can point out Baladi's paper [1], which gives a good account of how the theory on Gibbs' and equilibrium states can be carried over to finitely presented systems.

The questions treated here arise in a natural way from Markov partitions on Axiom A systems (cf. Smale [13] and Bowen [2]), for it is well known that an Axiom A diffeomorphism on some manifold M is semiconjugate to the shift on a subshift of finite type constructed by partitioning M in a certain way. However, as a subshift of finite type can be isomorphic only to an Axiom A diffeomorphism over a non-wandering set of zero dimension, the "boundary set", that is the set of points whose preimages in the shift consist of more than one point, contains essential in-

Received November 17, 1989 and in revised form October 15, 1990

formation about the topological structure of the non-wandering set, despite the fact that it has measure zero for any ergodic measure which is positive on open sets.

The present note resolves the question of canonical coordinates for finitely presented systems from the purely symbolic point of view, by establishing a combinatorial criterion (Theorem 9) as a necessary and sufficient condition for the existence of a local product structure. In the first section we introduce finitely presented systems as quotients of subshifts Σ_A. In section 2 we describe a non-transitive subshift which is reminiscent of the ones used by Manning [11]. The subshift thus obtained has a partial ordering (by inclusion) such that maximal elements correspond to equivalence classes in Σ_A. In section 4 we reduce the original Σ_A to a subshift in which transitive points have no other equivalent points besides themselves with respect to the induced equivalence relation. In section 5 we prove the main result, determining the conditions under which one can define a local product structure (Definition 7) on the quotient space Ω. Such a product structure is equivalent to local canonical coordinates given by the foliations of transversally intersecting stable and unstable directions. Let us note that Fried ([9], Lemma 3) constructs a finite-to-one extension of Ω with canonical coordinates.

1. Definitions

We consider a finite set of n symbols $\boldsymbol{A} = \{1, \ldots, n\}$ with the discrete topology. Let A be an irreducible $n \times n$-matrix of zeros and ones and define the shift transformation on the space

$$\Sigma_A = \left\{ x \in \prod_{i \in \mathbf{Z}} \boldsymbol{A} : A_{x_i, x_{i+1}} = 1,\ i \in \mathbf{Z} \right\}$$

by $(\sigma x)_i = x_{i+1}$, $i \in \mathbf{Z}$. We use the notation $a \to b$ if $A_{a,b} = 1$. We write $x_k \cdots x_l \in \Sigma_A$ if $x_k \cdots x_l$ is an allowed sequence in Σ_A and say $\tau \in \Sigma_A$ is a *loop* if $\tau\tau \in \Sigma_A$ and denote by τ^k its concatenation $\tau\tau \cdots \tau$, k-times, k a positive integer. The topology on Σ_A is generated by the cylinders sets

$$U(x_k \cdots x_l) = \{ y \in \Sigma_A : y_k \cdots y_l = x_k \cdots x_l \},$$

where $x_k \cdots x_l$ are finite words in Σ_A. The cylinders $U(x_k \cdots x_l)$ are closed-open sets in Σ_A, which is a totally disconnected space. A point x in Σ_A is *transitive* if for every $y \in \Sigma_A$ and $m > 0$ there exists an integer k such that $(\sigma^k x)_i = y_i$ for $|i| \le m$. The subshift Σ_A is (doubly) transitive if for every positive integer N there exist $n, n' \ge N$ such that $B \cap \sigma^n B' \neq \emptyset$, $B \cap \sigma^{-n'} B' \neq \emptyset$ for open and non-empty

$B, B' \subset \Sigma_A$. This is the same as that A is irreducible, or that for every $i,j \in \{1,\ldots,n\}$ there exists an $m > 0$ such that $(A^m)_{i,j} > 0$. Moreover, Σ_A is topologically mixing if for every open non-empty $B, B' \subset \Sigma_A$ there exists an N such that $B \cap \sigma^n B' \neq \emptyset$ for all $n \geq N$. Equivalent to this is that A^m is a positive matrix for all large enough m.

A finitely presented system is given as the quotient of a subshift of finite type Σ_A by an equivalence relation which is induced by a symmteric and reflexive relation on the alphabet $\boldsymbol{A}$. To begin with denote by $\sim$ a relation on $\boldsymbol{A}$ such that $a \sim a$, and $a \sim b$ implies $b \sim a$, $a, b \in \boldsymbol{A}$. We extend this relation to the subshift Σ_A and say $x \approx y$ if $x_i \sim y_i$ for all $i \in \mathbf{Z}$. If $\approx$ is transitive, that is $x \approx y \approx z$ implies $x \approx z$ for any three points $x, y, z \in \Sigma_A$, then $\approx$ is an equivalence relation on Σ_A. We say two words $x_k \cdots x_l, y_k \cdots y_l \in \Sigma_A$ are related, $x_k \cdots x_l \sim y_k \cdots y_l$, if $x_i \sim y_i$, $k \leq i \leq l$. Put $\alpha = |\boldsymbol{A}|^3$ and we have the following result.

Lemma 1. *A relation $\sim$ on $\boldsymbol{A}$ induces an equivalence relation on Σ_A in the above manner if and only if $x_0 \sim z_0$ for any three Σ_A-words $x_{-\alpha} \cdots x_\alpha \sim y_{-\alpha} \cdots y_\alpha \sim z_{-\alpha} \cdots z_\alpha$ related in the way indicated.*

Proof. If $x_0 \sim z_0$ for any three strings as in the Lemma, then $\approx$ is an equivalence relation by shift invariance.

Secondly, suppose $\sim$ induces an equivalence relation $\approx$ on Σ_A, and assume there are three words of length $2\alpha + 1$, $x_{-\alpha} \cdots x_\alpha \sim y_{-\alpha} \cdots y_\alpha \sim z_{-\alpha} \cdots z_\alpha$ related to each other in the given order for which $x_0 \nsim z_0$. We shall contradict the transitivity of $\approx$. The strings are sufficiently long so that $(y_k, x_k, z_k) = (y_l, x_l, z_l)$ for some $0 \leq k < l \leq \alpha$. Iterating this loop yields three points which are equivalent on positive coordinates. The same argument applied to negative indices yields three points $x, y, z \in \Sigma_A$ for which $x \approx y \approx z$ holds, but not $x \approx z$ since by assumption $x_0 \nsim z_0$. ∎

We immediately can make the following observation:

Corollary 2. *If $\approx$ is an equivalence relation on Σ_A, then $x_i \sim z_i$, $k + \alpha \leq i \leq l - \alpha$ for any three strings $x_k \cdots x_l \sim y_k \cdots y_l \sim z_k \cdots z_l$ for which $k + 2\alpha < l$.*

Assume that $\approx$ is an equivalence relation and denote by π the quotient map $\Sigma_A \to \Omega = \Sigma_A/\approx$. Then we call the quotient space Ω, which is equipped with the automorphism induced by σ and a suitable topology (section 3), a *finitely presented dynamical system*. We also say the pair $(\Sigma_A, \approx)$ is finitely presented.

In [9] Fried gave four equivalent characterizations of finitely presented systems, two more of which we shall mention here. A homeomorphism $T: \Omega \to \Omega$, Ω a compact topological space, is called *expansive* if there exists a closed neighborhood $V \subset \Omega \times \Omega$ of the diagonal Δ such that $T^* = T \times T : \Omega \times \Omega \to \Omega \times \Omega$ satisfies $\Delta = \bigcap_{-\infty<j<\infty} T^{*j} V$. We have the following result by D. Fried.

THEOREM 3 ([9], Theorem 1 and 2). *(Ω, T) is a finitely presented dynamical system if and only if either*

(i) *$T : \Omega \to \Omega$ is expansive, or*

(ii) *(Ω, T) admits arbitrarily fine but finite Markov partitions.*

A Markov partition of (Ω, T) is a partition of Ω into finitely many proper sets R_j which satisfy the so-called Markov property (R_j is a proper set if it is the closure of its interior). For a precise definition and details see [3] and [12].

Also note that Dateyama [6] showed that finitely presented systems $T : \Omega \to \Omega$ have the special pseudo-orbit tracing property, which means that Ω has a finite partition D such that any δ-pseudo-orbit $\{\xi_i : i\}$ ($d(T(\xi_i), \xi_{i+1}) < \delta$ for all i, δ small), for which $T(\xi_i)$ and ξ_{i+1} are in the same element of D for all times i, can be shadowed by a genuine orbit. In the special case where $D = \{\Omega\}$, Ω has the usual pseudo-orbit tracing property and thus canonical local coordinates. For subshifts it was shown by Walters [14] that the pseudo-orbit tracing property is equivalent to the subshift being of finite type.

The introduction of finitely presented systems in [9] was motivated by the desire to unify the existing theories of strongly hyperbolic systems (Axiom A), zero dimensional dynamical systems (sofic systems [15],[5]) and Thurston's pseudo-Anosov homeomorphisms. Let us now point out the connection with the first mentioned class, or, as we shall actually do, with Ruelle's Smale spaces [12]. Let (Ω, T) be a Smale space (a compact metric space with an expanding homeomorphism and a local product structure) with metric $d(\cdot,\cdot)$ and homeomorphism T with expansive constant ϵ. Let $\{R_j : j \in J\}$, J a finite index set, be a Markov partition of Ω, where the rectangles R_j are proper and satisfy $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ if $i \neq j$. We denote by A the associated transition matrix defined by $A_{i,j} = 1$ whenever $\operatorname{int} R_i \cap T^{-1}(\operatorname{int} R_j)$ is non-empty and zero otherwise. The Boolean matrix A defines a subshift Σ_A over the alphabet $\{1, \ldots, |\mathbf{J}|\}$. The shift $\sigma : \Sigma_A \to \Sigma_A$ is semi-conjugate to T, $T\pi = \pi\sigma$, where the projection $\pi : \Sigma_A \to \Omega$, given by $\pi(x) = \bigcap_{-\infty<i<\infty} T^{-i}(R_{x_i})$, is finite to one and one to one almost everywhere with respect to any ergodic measure positive on open sets and in particular on doubly transitive points (for definition see section 4) provided the partition is fine enough. If ∂R denotes the collective boundary set of the rectangles R_i then $\bigcup_{i\in\mathbf{Z}} T^i(\partial R)$ is precisely the subset on Ω on which π^{-1} is not unique. We say that two rectangles are related if they have non-empty intersection. It follows that $\approx$ is an equivalence relation. Here we used expansiveness, since for two equivalent points $x \approx y$, $d(T^k\pi(x), T^k\pi(y)) \le \epsilon$ for all integers k, and thus their π-images must coincide, $\pi(x) = \pi(y)$, if the rectangles are in diameter less than half an expansive constant ϵ.

2. The shift over related symbols

In this section we introduce a non-transitive subshift which plays the key rôle in discussing the quotient $\Sigma_A/\approx$. We shall assume that Σ_A is topologically mixing, i.e. $A^n > 0$ for n large enough.

From now on we assume that $\approx$ is an equivalence relation on Σ_A. Two words $x_k \cdots x_l, y_k \cdots y_l \in \Sigma_A$ are said to form a *diamond* if $x_k = y_k$, $x_l = y_l$ and a *collapsing diamond* if they are related. We will assume that $\approx$ does not collapse diamonds. This condition is in particular satisfied in the case where Σ_A is the subshift derived from a fine enough Markov partition of an Axiom A diffeomorphism. In [2] chapter 2 this argument is used to show that the quotient map $\pi : \Sigma_A \to \Omega$ is bounded (at most $|\boldsymbol{A}|^2$) to one. (To decide whether $\approx$ collapses diamonds it is enough to check pairs of strings whose length is at most $|\boldsymbol{A}|^2 + 1$.)

Denote by $\boldsymbol{A}_n$ the (unordered) subsets $\{a_0, \ldots, a_n\} \subset \boldsymbol{A}$ of $n+1$ pairwise related symbols none of which appears twice. (In case of a Markov partition, $\boldsymbol{A}_n$ consists of all combinations of $n+1$ neighboring rectangles.) We introduce an ordering on $\boldsymbol{A}_n$ by $\xi \le \zeta$, $\xi = \{a_0, \ldots, a_n\}$, $\zeta = \{b_0, \ldots, b_n\}$ if there is a set of related Σ_A-words $x_0^i \cdots x_m^i$, $i = 0, \ldots, n$, connecting ξ with ζ (i.e. $x_k^i \sim x_k^j$, $i,j = 0, \ldots, n$, $k = 0, \ldots, m$ for some $m \ge 1$, such that $\xi = \{x_0^i : i\}$, $\zeta = \{x_m^i : i\}$). Thus $\boldsymbol{A}_n$ is a disjoint union of subsets $\boldsymbol{A}_{n,k}$, $k = 1, \ldots, k_n$, such that if $\xi_0, \xi_1 \in \boldsymbol{A}_{n,k}$ then $\xi_0 \le \xi_1$, $\xi_1 \le \xi_0$ (and therefore $\xi_0 \le \xi_0$) unless $\boldsymbol{A}_{n,k}$ consists of a single symbol which cannot be repeated. If $\xi \in \boldsymbol{A}_{n,k}$ and $\zeta \in \boldsymbol{A}_{m,l}$, $(n,k) \ne (m,l)$, can be joined up, then either $\xi < \zeta$ or $\zeta < \xi$ (where $<$ means $\le$ holds but not $\ge$). In this way we partially order $\bigcup_{n,k} \boldsymbol{A}_{n,k}$. Observe that it is impossible to have $\xi_0, \xi_1 \in \boldsymbol{A}_{n,k}$, $\zeta \in \boldsymbol{A}_{m,l}$, $(n,k) \ne (m,l)$, such that $\xi_0 \le \zeta \le \xi_1$. This is obvious for $n = m$ (from the definition); and if $n \ne m$ we have, by transitivity of $\Sigma_{n,k}$, $\xi_0 \le \zeta \le \xi_1 \le \xi_0$, which implies that there are related words beginning in ξ_0 and returning to it. A set of related words running through the loop $\xi_0 \le \zeta \le \xi_1 \le \xi_0$ induces on ξ_0 a permutation π, a power of which is the identity. Since ξ_0 and ζ have different cardinalities we would obtain a collapsing diamond. Summing up: for $n \ne m$ we have thus always either $\boldsymbol{A}_{n,k} < \boldsymbol{A}_{m,l}$, or $\boldsymbol{A}_{m,l} < \boldsymbol{A}_{n,k}$, or neither.

We prune away those $\boldsymbol{A}_{n,k}$ whose elements do not occur in doubly infinite sequences composed over $\bigcup_{k,n} \boldsymbol{A}_{n,k}$. Denote the new symbol set by $\boldsymbol{C}$ and define a $|\boldsymbol{C}| \times |\boldsymbol{C}|$-transition matrix A^* by setting $A^*[\xi, \zeta] = 1$, $\xi, \zeta \in \boldsymbol{C}$, whenever there exist Σ_A-words $\{a_i a_j : i,j\}$ (of length 2) such that $\{a_i : i\} = \xi$ and $\{a_j : j\} = \zeta$; and $A^*[\xi,\zeta] = 0$ otherwise. This defines a subshift Σ_C whose transition matrix C is, with suitably arranged $\boldsymbol{C}$,

$$\begin{bmatrix} A_1^* & 0 & \cdots & 0 \\ 0 & A_2^* & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & 0 & A_s^* \end{bmatrix}$$

$s \geq 1$, where the submatrices A_t^* are of square block upper triangular form (if suitably arranged)

$$\begin{bmatrix} A_{t,1} & * & \cdots & * \\ 0 & A_{t,2} & \cdots & * \\ \vdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & A_{t,r} \end{bmatrix}$$

for some $r = r_t \geq 1$. One of the submatrices A_t^* contains A somewhere in its diagonal. The transition matrices $A_{n,k}$ determine in the usual way subshifts $\Sigma_{A_{n,k}}$ over the alphabets $\boldsymbol{A}_{n,k}$. Call $\{\boldsymbol{A}_{n,k} : k,n\}$ and $\{\Sigma_{A_{n,k}} : k,n\}$ from now on by single indices: $\{\boldsymbol{C}_i : i\}$ and $\{\Sigma_i : i\}$. Observe:

(i) $\boldsymbol{C}$ is closed under intersections of its elements (as subsets of $\boldsymbol{A}$);

(ii) the subshifts Σ_i are topologically transitive unless they are empty (then $\boldsymbol{C}_i = \{\zeta\}$ such that $\zeta \not\sim \zeta$);

(iii) Denote by $\delta'(i)$ the cardinality of the elements of $\boldsymbol{C}_i$ as subsets of $\boldsymbol{A}$ and call $\delta(i) = \delta'(i) - 1$ the *dimension* of $\boldsymbol{C}_i$ (also $\delta(\xi) = \delta(\boldsymbol{C}_i) = \delta(i)$, $\xi \in \boldsymbol{C}_i$).

For $x \in \Sigma_A$ we denote by $\langle x \rangle = \{z \in \Sigma_A : z \approx x\} \in \Sigma_C$ the equivalence class of x and by $\langle x \rangle_i = \{z_i \in \boldsymbol{A} : z \in \langle x \rangle\}$ its i-th coordinate. On Σ_C we have a partial ordering by inclusion: $x \subset y$ if $x_i \subset y_i$ (as subsets of $\boldsymbol{A}$) for all $i \in \mathbf{Z}$, $x,y \in \Sigma_C$. Equivalence classes $\langle x \rangle$ are maximal elements in Σ_C and, vice-versa, maximal elements in Σ_C correspond to points in the quotient $\Sigma_A/\approx$.

3. The topology on $\Sigma_A/\approx$

The natural class of Hölder equivalent metrics on $\Omega = \Sigma_A/\approx$ was determined in [8] by Fried. We outline his argument here. As in Corollary 2 let $\alpha = |\boldsymbol{A}|^3$ and define symmetric neighborhoods of the diagonal in $\Sigma_A \times \Sigma_A$ by

$$U_n = \{(x,y) \in \Sigma_A \times \Sigma_A : x_i \sim y_i \text{ for all } |i| \leq 2\alpha n\},$$

$n = 1, \ldots$. We claim that $U_n \circ U_n \circ U_n \subset U_{n-1}$, for all $n > 1$, where $U_n \circ U_n = \{(x,z) : \exists y \in \Sigma_A$ such that $(x,y),(y,z) \in U_n\}$. To see this, let $w,x,y,z \in \Sigma_A$ satisfy $w_i \sim x_i \sim y_i \sim z_i\ \forall |i| \leq 2\alpha n$. Corollary 2 applied once yields $w_i \sim y_i$ for $|i| \leq \alpha(2n-1)$ and a second time, shows that $w_i \sim z_i$, $|i| \leq 2\alpha(n-1)$. Hence $U_n \circ U_n \circ U_n \subset U_{n-1}$, $n > 1$, and by Frink's metrization lemma ([10], p. 185) there exists a pseudo-metric d on Ω, with the property $U_n \subset \{(x,y) : d(x,y) < 2^{-n}\} \subset U_{n-1}$, $n \in \mathbf{N}$. In fact d is a metric since $x,y \in \Sigma_A$ represent the same point in Ω if and only if $x_i \sim y_i$, $i \in \mathbf{Z}$, which is the case if and only if (x,y) lies in the intersection $\bigcap_{n\geq 1} U_n$, which is the diagonal in $\Sigma_A/\approx \times \Sigma_A/\approx$.

The distance function $d'(x,y) = \lambda^p$, where $p = \max\{q : x_i \sim y_i, |i| \leq q\}$ is with $\lambda = 2^{-2\alpha}$ equivalent to $d : C^{-1}d \leq d' \leq d$, where $C = 2^{2\alpha}$. As σ is expansive, the topology on Ω induced by d is generated by the "cylinder sets" $\pi(U(v_k \cdots v_l))$, where $U(v_k \cdots v_l) = \{z \in \Sigma_A : z_i \in v_i,\ k \leq i \leq l\}$ and $v_k \cdots v_l$ are finite strings in Σ_C. For $x \in \Sigma_A$ we define:

$$W^s(x,k) = \{z \in \Sigma_A : z_i \sim x_i,\ i \geq -k\}, \qquad W^u(x,k) = \{z \in \Sigma_A : z_i \sim x_i,\ i \leq k\},$$

whose unions over $k \in \mathbf{Z}$ are the stable, $W^s(x)$, and unstable, $W^u(x)$, directions through x. For $y \in W^s(x,1)$ we have $d(\sigma^{2l\alpha}x, \sigma^{2l\alpha}y) \leq 2^{-l}d(x,y)$, $l \geq 1$, and therefore $d(\sigma^l x, \sigma^l y) \leq C2^{-l/2\alpha}d(x,y)$, $l \geq 1$, with some $C \leq 2^{2\alpha-1}$. Following Mather we can replace this metric by an adapted one, d'', for which the constant C equals one: Let $\gamma = 2^{-1/2\alpha} < 1$ and define $d''(x,y) = \sum_{0\leq l<2\alpha} \gamma^{-l} d(\sigma^l x, \sigma^l y)$ which, as one readily verifies, is an adapted metric on Ω. The shift σ on Σ_A induces a homeomorphism on Ω which we again denote by σ. For positive k the homeomorphism σ on $W^s(x,k)$ contracts distances in the d''-metric by γ and σ^{-1} contracts distances on $W^u(x,k)$ by a factor γ. The stable and unstable directions through the points of Ω are $W^s(x,1)$ and $W^u(x,1)$. Interestingly enough, it is at this point possible to draw conclusions as to what the topological dimension of the quotient space might be. In fact, with a result of Fathi's ([7], Corollary 5.3), we get an upper bound. Hence, the topological dimension, which is bounded by the Hausdorff dimension $\mathrm{HD}_d(\Omega)$, is less than or equal to $2h(\Omega)/|\log\gamma| \leq 4\alpha h(\Sigma_A)/\log 2$, where h is the topological entropy. This estimate applies to any α for which the statement of Lemma 1 holds, and in general can be chosen much smaller than $|A|^3$.

4. Reducing Σ_A

The condition that equivalence classes are finite does not suffice to guarantee that a transitive point has no other equivalent point besides itself. In this section we introduce a reduction procedure and show that it is always possible to assume that transitive points have trivial equivalence classes. A point x in Σ_A (Σ_l) is (doubly) *transitive* if for any given $y \in \Sigma_A$ (Σ_l) and $n \geq 1$ one can find positive integers $m, \hat{m}$, such that $y_i = (\sigma^m x)_i = (\sigma^{-\hat{m}} x)_i$ for $|i| \leq n$. In other words, every Σ_A-word (Σ_l-word) appears infinitely often in the past and future dimensions of x. In this section we treat the case where Σ_A has transitive points with non-trivial equivalence classes and by using Theorem 5 we shall see that Σ_A can be replaced by another subshift of finite type in which transitive points have trivial equivalence classes and whose quotient is isomorphic to $\Omega = \Sigma_A/\approx$.

Similar to the notation we introduced for Σ_A we set $U(\chi_s \cdots \chi_t) = \{\xi \in \Sigma_l : \xi_s \cdots \xi_t = \chi_s \cdots \chi_t\}$, $\chi_s \cdots \chi_t \in \Sigma_l$, for the closed-open cylinders in Σ_l with $\chi_s \cdots \chi_t$ on the coordinates from s to t.

LEMMA 4. *Let χ be a transitive point in some Σ_l. Then any two $\xi, \zeta \in \chi \subset \Sigma_A$ are either identical or disagree on all coordinates.*

PROOF. Let χ be a transitive point in Σ_l. We have to show that different Σ_A-points $\xi, \zeta \in \chi$ differ on all places, that is $\xi_i \neq \zeta_i$ for all $i \in \mathbf{Z}$. Suppose $\xi_k = \zeta_k$ for some k and pick an l such that $\xi_l \neq \zeta_l$. Without loss of generality we can assume that $k < l$. Since χ is transitive, the word $\chi_k \cdots \chi_l$ appears infinitely often, on the "positive side", say at intervals of lengths $m_1, m_2, \ldots$ (all of which we assume to be bigger than $l - k$). We want to construct a collapsing diamond. Now $\chi_k \cdots \chi_l \cdots \chi_{k+m_1}$ need not have a collapsing diamond in Σ_A, because strings beginning in χ_k (as subsets of $\boldsymbol{A}$) at the same element do not necessarily end up again at the same element in $\chi_{k+m_1} = \chi_k$. However, it follows from Cartan's drawer principle that $\chi_{k+m_p} \cdots \chi_{k+m_q}$ has a collapsing Σ_A-diamond for some $p < q$. ■

Put $\Omega_k = \Sigma_k/\approx$, where we denote by $\approx$ the equivalence relation induced on Σ_C. For k, l satisfying $\delta(k) = \delta(l) + 1$, define a map v from $\boldsymbol{C}_k$ into $\boldsymbol{C}_l^*$, the power set of C_l, by

$$v(\chi) = \{\{\chi^1, \ldots, \chi^{\delta(k)}\}, \{\chi^0, \chi^2, \ldots, \chi^{\delta(k)}\}, \ldots, \{\chi^0, \chi^1, \ldots, \chi^{\delta(k)-1}\}\} \cap \boldsymbol{C}_l,$$

$\chi = \{\chi^0, \ldots, \chi^{\delta(k)}\} \in \boldsymbol{C}_k$ and put $v(\boldsymbol{C}_k) = \bigcup\{v(\chi) : \chi \in \boldsymbol{C}_k\} \subset \boldsymbol{C}_l$. In general only the inverse v^{-1} is well defined on $v(\boldsymbol{C}_k)$. Denote by $v(\Sigma_k)$ the subshift over the alphabet $v(\boldsymbol{C}_k)$ with transitions induced by v. In other words, we set $\zeta \to \zeta'$ if $(\zeta, \zeta') \in v(\chi) \times v(\chi')$ for some $\chi, \chi' \in \boldsymbol{C}_k$, $\chi \to \chi'$. If $\varnothing \neq v(\boldsymbol{C}_k) \subset \boldsymbol{C}_l$, then $v(\Sigma_k)$ is the "union" of subshifts that are isomorphic to each other. We formulate the main result of this section.

THEOREM 5. (i) *If* $v(\Sigma_k) = \Sigma_l$ *then* $\Omega_k \cong \Omega_l$, *and*

(ii) *if* $v(\Sigma_k) \neq \Sigma_l$ *for all* Σ_k *with* $\delta(k) = \delta(l) + 1$, *then transitive points in* Σ_l *have trivial equivalence classes.*

PROOF. It is obvious that equivalence classes in Σ_k are by $v : \Sigma_k \to \Sigma_l$ again mapped into equivalence classes, and, on the other hand, it is easily seen that points in Σ_k that are not equivalent cannot be mapped to equivalent points in Σ_l. Hence if $v(\Sigma_k) = \Sigma_l$, then their quotients are isomorphic, where the isomorphism is defined in the obvious way by v.

We show that if every point in Σ_l has a non-trivial equivalence class then there exists a Σ_k such that $v(\Sigma_k) = \Sigma_l$. Suppose ξ is a sequence in Σ_l with non-trivial equivalence class. By Lemma 4 it follows that $\xi_i \notin \zeta_i$ for all $\zeta \in \langle \xi \rangle \backslash \xi \subset \Sigma_A$. Set $\xi_i = \{\xi_i^0, \ldots, \xi_i^{\delta(l)}\}$, $\zeta_i = \{\zeta_i^0, \ldots, \zeta_i^{\delta(l)}\}$. Then we can find indices $j : \mathbf{Z} \to \{0, \ldots, \delta(l)\}$ so that $\zeta_i^{j(i)} \notin \xi_i$ and $\zeta_i^{j(i)} \to \zeta_{i+1}^{j(i+1)}$, $i \in \mathbf{Z}$. Thus $\xi_i' = \xi_i \cup \zeta_i^{j(i)}$ are elements in some $\boldsymbol{C}_k$, $\delta(k) = \delta(l) + 1$. In particular ξ can be chosen to be transitive and therefore realises every possible transition. Thus $\xi' \in \Sigma_k$ and $v(\xi') = \xi$ from which follows that $v(\Sigma_k) = \Sigma_l$ and therefore by the first part of the theorem $\Omega_k \cong \Omega_l$. ■

DEFINITION 6. We call a subshift Σ_l or subalphabet C_l *reduced* if Σ_l satisfies the condition of Theorem 5(ii).

As a consequence of Theorem 5, for any finitely presented system $(\Sigma_A, \approx)$, there are numbers $\ell[0], \ldots, \ell[p]$ such that

(i) $\delta(\ell[q+1]) = \delta(\ell[q]) + 1$, $v(\Sigma_{\ell[q+1]}) = \Sigma_{\ell[q]}$ for $0 \leq q < p$,

(ii) transitive points in $\Sigma_{\ell[p]}$ have trivial equivlaence classes, and

(iii) $\Omega_{\ell[q]} \cong \Sigma_A / \approx$, $0 \leq q \leq p$.

5. The product structure on Ω

In this section we prove the main result, giving necessary and sufficient conditions under which a finitely presented system has a local product structure. The criterion appears rather natural. The idea is to consider half-infinite related strings and to examine the situations under which a positive infinite sequence can be linked to a negative infinite sequence. In practice it takes a finite number of steps to decide this problem, which makes Theorem 9 easily accessible to the actual computation of concrete examples. We use the definition of product structure as given by Ruelle ([12], chapter 7).

DEFINITION 7. A *local product structure on* Ω is a map $[\cdot, \cdot] : \Omega \times \Omega \to \Omega$ defined in a neighborhood of the diagonal and has the properties

(i) $[x,x] = x$, $[[x,y],z] = [x,[y,z]] = [x,z]$, $[\sigma x, \sigma y] = \sigma[x,y]$, whenever defined;

(ii) there exist $\gamma > 0$, $\lambda \in (0,1)$ so that

(α) if $d(y_i,x) < \gamma$ and $[y_i,x] = y_i$, $i = 1,2$, then $d(\sigma^n y_1, \sigma^n y_2) \leq \lambda^n d(y_1,y_2)$, $n > 0$;

(β) if $d(x,z_i) < \gamma$ and $[x,z_i] = z_i$, $i = 1,2$, then $d(\sigma^n z_1, \sigma^n z_2) \leq \lambda^{|n|} d(z_1,z_2)$, $n < 0$.

The local stable and unstable directions through x are given by

$$W^s_{\mathrm{loc}}(x) = \{y \in \Omega : [x,y] = y,\ d(x,y) \leq \lambda\}$$

and

$$W^u_{\mathrm{loc}}(x) = \{y \in \Omega : [y,x] = y,\ d(x,y) \leq \lambda\}.$$

The point $[x,y]$ lies in the stable direction of x and in the unstable direction of y. The product structure on subshifts is given by $[x,y] = \cdots y_{-1}y_0x_1x_2\cdots$ and defined whenever $d(x,y) < 1$ (if the subshift is of type 2) with the usual metric introduced in section 1. Let us note that Bowen ([4], Proposition 6.2) showed that if a shiftspace has a local product, the subshift necessarily is of finite type.

We have to suffer some more notation and begin by introducing one-sided subshifts (as usual $A^n > 0$, for n large enough):

$$\Sigma_{A^-} = \left\{x \in \prod_{-\infty,\dots,0} \boldsymbol{A} : A_{x_i,x_{i+1}} = 1\ \forall i < 0\right\},$$

$$\Sigma_{A^+} = \left\{y \in \prod_{0,\dots,+\infty} \boldsymbol{A} : A_{y_i,y_{i+1}} = 1\ \forall i \geq 0\right\}.$$

Denote by $U^+(a)$ the cylinder that consists of all sequences $y_0y_1\cdots \in \Sigma_{A^+}$ which begin with $y_0 = a$ and similarly $U^-(a) = \{\cdots x_{-1}x_0 \in \Sigma_{A^-} : x_0 = a\}$. Two sequences $x_0x_1\cdots$, $y_0y_1\cdots$ in Σ_{A^+} are related, $x_0x_1\cdots \sim y_0y_1\cdots$, whenever $x_i \sim y_i$, $i = 0,1,\dots$ (similarly for Σ_{A^-}). We use the notation $\pm$ whenever either sign applies. For $\zeta \in \Sigma_{A^\pm}$ we put

$$S^\pm(\zeta) = \{\xi \in \Sigma_{A^\pm} : \xi \sim \zeta\},$$

for the set of one-sided sequences related to ζ. Denote by π_i the projection onto the i-th coordinate, for instance $\pi_0 S^-(\zeta) = \{y_0 : \cdots y_{-1}y_0 \in S^-(\zeta)\}$, similarly $\pi_0 S^+(\zeta)$. As $\boldsymbol{C}$ and Σ_C the corresponding one-sided objects $\boldsymbol{C}^+, \boldsymbol{C}^-, \Sigma_{C^+}, \Sigma_{C^-}$ are defined as follows. To find $\boldsymbol{C}^+$ we take $\bigcup_{k,n} \boldsymbol{A}_{n,k}$ and prune away those symbols that cannot (in Σ_C) be extended infinitely in a forward direction. Similarly $\boldsymbol{C}^-$ is found as $\bigcup_{k,n} \boldsymbol{A}_{n,k}$ less those elements that do not allow an infinite backward extension. Transition matrices C^+, C^- (and similarly $A^+(i), A^-(i)$) are defined in the same way as in section 2. We call the associated one-sided shift spaces Σ_{C^+}, Σ_{C^-}. Let $\beta = 2^{|\boldsymbol{A}|}$ and note that (i) $\boldsymbol{C}^+ \cap \boldsymbol{C}^- = \boldsymbol{C}$, (ii) $\Sigma_{C^+}, \Sigma_{C^-}$ have the same transitive subshifts as Σ_C, and (iii) if $u_0 u_1 \cdots \in \Sigma_{C^+}$, then $u_\beta u_{\beta+1} \cdots$ is a half infinite sequence in Σ_C, and if $\cdots u_{-1} u_0 \in \Sigma_{C^+}$, then $\cdots u_{\beta-1} u_\beta$ is a one-sided sequence in Σ_{C^-}.

Analogously to the two-sided case the elements in $\Sigma_{C^\pm}$ are partially ordered by inclusion. Given $\zeta \in \Sigma_{C^\pm}$ then $S^\pm(\zeta)$ are the maximal elements in $\Sigma_{C^\pm}$ that contain ζ and we put M^+ for all $u_0 \in \boldsymbol{C}^+$ for which one can find a sequence $u_0 u_1 \cdots$ maximal in Σ_{C^+}, in other words $M^+ = \{\pi_0 S^+(\zeta) : \zeta \in \Sigma_{A^+}\}$; similarly $M^- = \{\pi_0 S^-(\zeta) : \zeta \in \Sigma_{A^-}\}$. Let $M \subset \boldsymbol{C}$ be the elements which are used to compose maximal strings in Σ_C. Obviously

$$M = \{u \cap v : (u,v) \in M^- \times M^+\} \setminus \{\emptyset\}.$$

Set $M^\pm(a) = \{u \in M^\pm : a \in u\}$, for $a \in \boldsymbol{A}$.

For a given $w = \{a,b\} \in \boldsymbol{C}$, we define *relative cylinders* $U^\pm(a \mid b)$ as follows:

$$U^+(a \mid b) = \{\zeta \in U^+(a) : \exists \xi \in S^+(\zeta) \cap U^+(b),$$
$$\{a,b\} = \{\zeta_0, \xi_0\} \to \cdots \to \{\zeta_s, \xi_s\} \leq \{\zeta_s, \xi_s\},\ s \geq 0\}.$$

The sequence of symbols $\{\zeta_j, \xi_j\}$ originating at $\{a,b\}$, which is not necessarily part of a transitive subsystem, leads into some transitive subshift (containing $\{\zeta_s, \xi_s\}$). If $\{a,b\}$ were a symbol in some transitive subshift Σ_i, then $U^+(a \mid b)$ would simply be $U^+(a)$. In the same manner one defines $U^-(a \mid b) \subset U^-(a)$ and

$$S^\pm(\zeta \mid b) = S^\pm(\zeta) \cap U^\pm(a \mid b), \qquad M^\pm(a \mid b) = \{\pi_0 S^\pm(\zeta \mid b) : \zeta \in U^\pm(a \mid b)\},$$

$\zeta \in U^\pm(a \mid b)$. Clearly one has $a, b \in u$ for all $u \in M^\pm(a \mid b)$.

DEFINITION 8. We call $w = \{a,b\} \in \boldsymbol{C}$ *isolated* if at least one of the intersections $u \cap v : (u,v) \in M^\mp(a \mid b) \times M^\pm(b \mid a)$ is empty.

We are able to formulate the main result.

THEOREM 9. *Ω has a local product structure if and only if $\boldsymbol{C}$ has no isolated elements.*

The result will follow from Lemmas 10 and 11. The next lemma shows the isolated elements make it impossible to define the local product. We will find a point in Ω and pairs of points converging to it in the Ω-topology, but whose local stable and unstable "leaves" have empty intersection, thus contradicting the uniformity property.

Lemma 10. *There is no local product structure on Ω if $\mathbf{C}$ has an isolated element.*

Proof. We shall construct a sequence of pairs $(x^q, y^q) \in \Sigma_A \times \Sigma_A$ that converge to a point in the diagonal of $\Omega \times \Omega$ and has the property that the local stable direction through x^q and the local unstable direction through y^q have empty intersection. (We write $[x,y]$ instead of $\langle[\langle x\rangle,\langle y\rangle]\rangle$ as we ought to.) Denote by $U_\gamma(x) \subset \Omega$ the ball of radius γ in the d-metric centered at $\langle x\rangle \in \Sigma_A/\approx$. We shall see that for any positive γ we cannot find $\epsilon > 0$ so that, although $d(x,y) \le \epsilon$, the local stable leave, $W^s(y) \cap U_\gamma(y)$, and the local unstable leave, $W^u(x) \cap U_\gamma(x)$, have non-empty intersection. In particular we will arrange that (x^q, y^q) converges exponentially fast to the diagonal in $\Omega \times \Omega$ as $d(x^q, y^q) \le 2^{[qr/2\alpha]}$ $(r \ge 1)$ for $q \to \infty$, while all the intersections $W^u(x^q, \beta) \cap W^s(y^q, \beta)$ are empty.

Let $w = \{a, b\}$ be isolated and say $u \cap v = \varnothing$ for some $(u, v) \in M^-(a \mid b) \times M^+(b \mid a)$. Hence there are sequences $(\cdots \xi_{-1}\xi_0, \zeta_0\zeta_1 \cdots) \in U^-(a) \times U^+(b)$ so that

$$\pi_0 S^+(\cdots \xi_{-1}\xi_0) = u, \qquad \pi_0 S^-(\zeta_0\zeta_1 \cdots) = v,$$

and by definition of $S^-(b \mid a)$, y_0 can be continued for negative coordinates by a Σ_A-word $\zeta_s \cdots \zeta_{-1}$, $\zeta_{-1} \to \zeta_0$, related to $\xi_s \cdots \xi_{-1}$ and such that $\{\xi_s, \zeta_s\} \le \{\xi_s, \zeta_s\}$ $(s < 0)$. Let $\{\nu_1', \tau_1'\} \to \cdots \to \{\nu_l', \tau_l'\} = \{\xi_s, \zeta_s\}$ be a Σ_C-loop. Similarly one attaches to ξ_0 on the positive side a Σ_A-word $\xi_1 \cdots \xi_t$, $\xi_0 \to \xi_1$, related to $\zeta_1 \cdots \zeta_t$, satisfying $\{\xi_t, \zeta_t\} \le \{\xi_t, \zeta_t\}$ and picks a Σ_C-loop $\{\nu_1'', \tau_1''\} \to \cdots \to \{\nu_k'', \tau_k''\} = \{\xi_t, \zeta_t\}$. For $q \in \mathbf{N}$ we define (bold characters indicate the zero position)

$$x^q = \cdots \xi_{s-1} \nu'^q \xi_s \cdots \boldsymbol{\xi}_0 \xi_1 \cdots \xi_t \nu''^q \cdots,$$

$$y^q = \cdots \tau'^q \xi_s \cdots \boldsymbol{\zeta}_0 \zeta_1 \cdots \zeta_t \tau''^q \zeta_{t+1} \cdots,$$

where the dots to the right of ν'' and to the left of τ' denote something that makes x^q and y^q, respectively, one-sided transitive. We show that the intersection $W^u(x^q, \beta) \cap W^s(y^q, \beta)$ is empty. Set

$$u^q = \pi_0 S^+(\xi_0 \cdots \xi_t \nu''^q \cdots) \subset u,$$

$$v^q = \pi_0 S^-(\cdots \tau'^q \zeta_s \cdots \zeta_0) \subset v,$$

and $u^q \cap v^q = \varnothing$, $q \ge 1$. Since by construction x^q and y^q are (one-sided) transitive, they cannot be equivalent (here we use the assumption that Σ_A is reduced). If there were a $z \in W^u(x^q,\beta) \cap W^s(y^q,\beta)$, the zeroth coordinate of it, z_0, would have to lie in the intersection $u^q \cap v^q$, which, however, is empty by assumption. On the other hand, one easily verifies that $d(x^q,y^q) \leq 2^{-[qr/2\alpha]}$ and dies exponentially fast as q goes to infinity, where $r = \min(k,l) \geq 1$. ∎

LEMMA 11. *If* $\boldsymbol{C}$ *has no isolated elements then* Ω *has a local product structure.*

PROOF. Put $\beta = 2^{|A|}$ and we shall describe how to construct $[y,x]$. If $d(x,y)$, $x,y \in \Sigma_A$, is small enough so that $x_i \sim y_i$, $|i| \leq \beta$, by Cartan's drawer principle at least one of the $\{x_i,y_i\}_{-\beta\leq i\leq\beta}$ appears twice and thus lies in some $\boldsymbol{C}_j$, say it is $\{x_0,y_0\}$. Put $u = \pi_0 S^-(\cdots x_{-1}x_0|y_0)$, $v = \pi_0 S^+(y_0y_1\cdots|x_0)$. Since $(u,v) \in M^-(x_0|y_0) \times M^+(y_0|x_0)$ and $\{x_0,y_0\}$ is not isolated, the intersection $u \cap v$ is not empty. Hence there exist half-infinite sequences

$$(\cdots x'_{-1}x'_0, y'_0y'_1\cdots) \in S^-(\cdots x_{-1}x_0|y_0) \times S^+(y_0y_1\cdots|x_0)$$

such that $x'_0 = y'_0 \in u \cap v$. It is now easily verified that

$$[y,x] = \cdots x'_{-2}x'_{-1}y'_0y'_1\cdots$$

satisfies the conditions for the local product structure given in Definition 6. To verify the identity $[[x,y],z] = [x,z]$ for $x,y,z \in \Sigma_A$ close enough to each other, one proceeds similarly. ∎

ACKNOWLEDGEMENTS

I acknowledge financial support by the I.H.E.S., where the first version of this paper was written. I am also grateful to the referee for having brought to my attention the paper by Dateyama [6].

REFERENCES

1. V. Baladi, *Gibbs and equilibrium states for finitely presented dynamical systems*, J. Stat. Phys., to appear.

2. R. Bowen, *On Axiom A Diffeomorphisms*, CBMS Reg. Conf. 35, American Mathematical Society, Providence, 1978.

3. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Math. **470**, Springer, Berlin, 1975.

4. R. Bowen, *Topological entropy and Axiom A*, Proc. Am. Math. Soc. **14** (1971), 23–42.

5. E. Coven and M. Paul, *Sofic systems*, Isr. J. Math. **20** (1975), 165–177.

6. M. Dateyama, *Homeomorphisms with Markov partitions*, Osaka J. Math. **26** (1989), 411–428.

7. A. Fathi, *Expansiveness, hyperbolicity and Hausdorff dimension*, Commun. Math. Phys. **126** (1989), 249–262.

8. D. Fried, *Métriques naturelles sur les espaces de Smale*, C.R. Acad. Sci. Paris **297** (1983), 77–79.

9. D. Fried, *Finitely presented dynamical systems*, Ergodic Theory & Dynamic Systems **7** (1987), 489-507.

10. J. Kelly, *General Topology*, Van Nostrand, Princeton, NJ, 1955.

11. A. Manning, *Axiom A diffeomorphisms have rational zeta functions,* Bull. London Math. Soc. **3** (1971), 215-220.

12. D. Ruelle, *Thermodynamic Formalism,* Addison Wesley, Reading, Mass., 1978.

13. S. Smale, *Differential dynamical systems*, Bull. Am. Math. Soc. **73** (1967), 747-817.

14. P. Walters, *On the pseudo orbit tracing property and its relationship to stability*, Lecture Notes in Math. **668**, Springer, Berlin, 1978, pp. 231-244.

15. B. Weiss, *Subshifts of finite type and sofic systems*, Monatsh. Math. **77** (1973), 462-474.